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Efficiency and Stability in a Discrete Model of Country Formation ⋆

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Abstract. This paper studies efficient and stable country configurations in a simple model of country formation. Driving force of the model is a trade-off between the benefits of large countries and the costs of heterogeneity of large and diverse populations. We show that efficient configurations as well as stable configurations exist for each value of the model parameter; however, there is no unambiguous relation between them. Moreover, country sizes in efficient configurations may differ by at most two, while in stable configurations the differences in their sizes may be relatively high. Our results contrast with those of Alesina and Spolaore (1997).

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1. Introduction

This paper studies efficient and stable country configurations in a simple model of country formation. Driving force of the model is a trade-off between the benefits of large countries and the costs of heterogeneity of large and diverse populations. Large jurisdictions bring several benefits with them. For example there may be economies of scale in public good provision or simply the fact that the (fixed) costs of government provision are shared among many citizens. On the other hand there is an opposite force. Since larger populations are less homogeneous, the choice of the type of government may become less similar to the individuals' most preferred type when the size of the country increases.

The focus of our model on the trade-off between benefits and costs of large groups is a frequent feature of the literature seeking to explain the formation of groups in societies see e.g. Demange (1994). For a detailed exposition of the benefits and costs concerning the organization of the world in countries see Alesina and Spolaore (1997) (henceforth AS) or Le Breton and Weber (2000).

Both our model and the questions we want to answer are inspired by AS. Alesina and Spolaore assume that the world population is one-dimensional and

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uniformly distributed on the unit interval. Individuals join together to form 'countries', that are intervals with the 'government' located in its middle. Each individual has the utility function of the same form, piecewise linear, directly proportional to the negative of its distance from the government and indirectly proportional to the negative of the size of its country. AS then consider two criteria for country formation: efficiency (the average utility of a citizen should be maximized) and stability (for precise definitions see the paper in question). They show that under efficiency as well as under stability criterion all countries are intervals of the same size. One of their main conclusions is that the efficient number of countries is always smaller than the stable one, from which they conlude that

> '... The democratic process leads to an inefficiently large number of countries. Namely, when coutries are formed through a democratic process, more countries are created than with a social planner who maximizes world average utility. ... '

Our aim is to test how reliable the conclusions of AS are. The main difference of our model and that of AS is that they work in a continuous setting and our model is discrete. Further, we slightly modified the stability notion to make a better sense for the discrete model. We show that for each value of the parameter determining the exact form of the utility function (more precisely, expressing the relative importance of its 'distance dependent' part to its 'country-size dependent' part) efficient configurations as well as stable configurations exist, however, there is no unambiguous relation between them. Moreover, although in an efficient configuration the country sizes are 'almost equal', in a stable one they may assume any integer value in a certain (not very small) interval.

One explanation of the differences obtained may be that in a discrete model the forces at play are changed considerably, since adding a player to a coalition alters substantially the coalition, whereas in the continuous model the measure (and the power) of a single player is zero. Is some situations however, a discrete model may be more appropriate. One reason may be that the real world is discrete and world population is large but not infinite. Another consideration may be that the formation of countries is driven by regions and a region may be a set of individuals with similar geografic position and utility (consider the disintegration of the Former Soviet Union, Yogoslavia, Czechoslovakia, as well as the movements for regional authonomy or even independence in e.g. Canada, Spain, France or Italy).

The first discretizations of AS have been attempted by Dahm (1999) and (2000). Dahm uses the same stability notions as AS and shows that in the discrete case a stable configuration may fail to exist. Moreover, in situations where a stable configuration exists, the relation to efficient configurations is not so clear as in AS.

Other papers dealing with relations between efficiency and stability of coalitional structures are Drèze and Greenberg (1980) and LeBreton and Weber (2000). In the first one the authors consider a model of an economy as a cooperative game where the utility function of each individual has two arguments: his consumption bundle and the coalition to which he belongs. An example is presented with three agents and two commodities where efficient allocations are not stable. The second paper focuses on one country and analyses when it is efficient to maintain a unified country as well as when this country is stable, that is, no region wants to secede. The paper shows that it is possible to reconcile both requirements.

The AS paper belongs to political economy literature on country formation which is reviewed in Bolton, Roland and Spolaore (1996). The AS model has recently been extended by Le Breton and Weber (2000). Unlike our paper, the latter focuses on group deviations. Another related work is Haeringer (2000). He also uses a discrete model, but deals only with stable structures. On the other hand, his model is more general than ours. However this generality makes it impossible to describe stable structures in more detail.

Our model can also be understood as a hedonic coalition formation game. This notion was introduced by Drèze and Greenberg (1980), who defined a hedonic game as such in which the utility of a member of a given coalition is not affected by the way outsiders organize themselves but depends only on the identity of the other members of her coalition. Recent contributions include Banerjee, Konishi and Sönmez (1998) and Bogomolnaia and Jackson (1998). These papers deal solely with various definitions of stable coalition structures and restrictions of the admissible preferences.

Inspired by the seminal work of Tiebout (1956) are the papers on local public good economies like Guesnerie and Oddou (1981), Greenberg and Weber (1986), (1993), Weber and Zamir (1986) and Jehiel and Scotchmer (1997). They mainly investigate the existence and characterization of stable partitions of individuals into jurisdictions rather than providing a detailed comparison of stable and efficient structures.

This paper is organized as follows. In Section 2 we present the basic model. Section 3 characterizes efficient configurations and Section 4 is devoted to our main stability notion which is slightly extended in Section 5. Section 6 concludes.

2. The Model

We suppose that the world population is finite, its cardinality will be denoted by W. Individuals are located in discrete points on the line, the distance between any two neighbouring locations is 1. Individuals join together in order to provide a (local) public good which is nonrival and excludable. We interpret the public good as ‘government’. It may represent a bundle of administrative, judicial, economic services and public policies. Each government identifies a country. (We will use from now on the terminology *country* for coalitions that form and *government* for the public good that must be provided.)

The cost of the public good provision is constant and is independent from the size of the country. Each individual has to belong exactly to one country. Because of excludability, the benefits arise only to citizens of a given country, who in turn have to finance their government.

The policy space we analyze is unidimensional. Each point in this interval represents a type of government and for each individual there exists a unique ideal policy. This allows us to identify each individual unambiguously with a point in the policy space. Hence we consider the individual's ideal point to be its 'location'.

Each individual has the same utility function, separable in the public good and money which takes the piecewise linear form similar to the one in AS

$$u(i) = u(1 - a\ell_i) + y - \frac{k}{|\mathcal{P}_i|}, \tag{2.1}$$

where u, a, k are nonnegative parameters, y is an exogeneous income of the individual (the same for all individuals), u measures the maximum utility derived when the location of individual coincides with the location of the government, ℓ_i is the distance of individual i from his government and $|\mathcal{P}_i|$ the number of citizens of the country of individual i.

Since the constant terms in the utility function are irrelevant, its maximization is equivalent to minimization of the individuals' cost function

$$w(i) = ua\ell_i + \frac{k}{|\mathcal{P}_i|}. \tag{2.2}$$

In order to make our treatment more concise, we shall consider the normalized cost

$$w_n(i) = c\ell_i + \frac{1}{|\mathcal{P}_i|}, \tag{2.3}$$

where the nonnegative parameter $c = \frac{ua}{k}$ measures the relative importance of the 'heterogeneity' costs with respect to the 'government provision' costs.

It will turn out that it is sufficient to consider 'connected countries'. We say that a country $\mathcal{P}$ is connected if $\mathcal{P}$ contains with any two citizens i, j all the intermediate citizens. Hence, a country $\mathcal{P}$ is connected if and only if the distance between any two neighbouring citizens of $\mathcal{P}$ is equal 1. Therefore we shall denote a configuration of countries as a vector $(p_1, p_2, \dots, p_N)$ of country sizes, assuming that the first country from the left has p_1 citizens, the second one p_2, etc.

We shall look at two different criteria for the formation of countries:

- *Efficiency.* How many countries should be created and of what size if the sum of costs of all the world inhabitants is to be minimized?
- *Stability.* We consider the following three possibilities for deviations:
  - A citizen leaves his original country and forms a new country of his own.
  - A citizen leaves one country to join a neighbouring country.
  - A citizen of one country and a citizen of a neighbouring country leave their countries to form a new country.

3. Efficiency

Let us call the sum of distances of all the citizens of a country $\mathcal{P}$ from its government the heterogeneity $H(\mathcal{P})$ of $\mathcal{P}$. The heterogeneity $H(\mathcal{E})$ of a configuration $\mathcal{E}$ is the sum of heterogeneities of all countries in $\mathcal{E}$. The minimum possible heterogeneity of a country with p inhabitants will be denoted by $h(p)$.

In the proof of the following assertions we shall call each citizen located to the left of the government of his country a *left citizen* and the one located to the right a *right citizen*. For a real number a, $\lfloor a \rfloor$ and $\lceil a \rceil$ denote the greatest integer not greater than a and the smallest integer not smaller than a, respectively.

LEMMA 1. *Let $\mathcal{P}$ be any country. Then the heterogeneity of $\mathcal{P}$ is minimal if its government is located at a median position, i.e. when the number of the left citizens of $\mathcal{P}$ is equal to the number of the right citizens of $\mathcal{P}$.*

Proof. Let $|\mathcal{P}| = p$ and let the government be located at a median position. We shall denote by $H^*(\mathcal{P})$ the heterogeneity of $\mathcal{P}$ and by p_L^* and p_R^* the number of the left and right citizens of $\mathcal{P}$. Clearly, $p_L^* = p_R^* = \lfloor \frac{p}{2} \rfloor$. Now, let the position of the government be w.l.o.g. in the distance x to the left of the original position. Let the heterogeneity of $\mathcal{P}$ be now $H'(\mathcal{P})$. Now, all the right citizens remain right, but each one of them has now his distance from the government increased by x. Of the left citizens some remain left (and the distance from the government of each one of them will be decreased by x) and some may be 'jumped' over. Let us denote the number of jumped-over citizens by j. If their distances from the government were originally $y, d_1 + y, d_2 + y, \ldots, d_{j-1} + y$ (where $y \in (0, d_1)$) from the rightmost one to the leftmost one, now they are $x - y, x - y - d_1, \ldots, x - y - d_{j-1}$ in the same order. So

$$H^*(\mathcal{P}) - H'(\mathcal{P}) = -p_R^* x + (p_L^* - j)x + j(2y - x) = 2j(y - x),$$

if the government was originally not located in any citizen and

$$H^*(\mathcal{P}) - H'(\mathcal{P}) = -p_R^* x + (p_L^* - j)x + j(2y - x) - x = 2j(y - x) - x$$

otherwise. Since $y - x < 0$ if $j \geqslant 1$ and $x > 0$, we have $H^*(\mathcal{P}) - H'(\mathcal{P}) \leq 0$, which implies the claim. □

Let us notice here that in case $|\mathcal{P}|$ is even, the minimum heterogeneity is achieved also if the government is located in anyone of middle citizens, which are not median positions according to the definition.

Moreover, the above theorem is true for any, not only uniform distances between citizens, but the next one already uses this uniformity.

LEMMA 2. *Let a country $\mathcal{P}$ have p citizens. Then $H(\mathcal{P}) = h(p)$ if and only if $\mathcal{P}$ is connected and in this case*

$$h(p) = \begin{cases} \frac{p^2}{4}, & \text{if } p \text{ is even;} \\ \frac{p^2-1}{4}, & \text{if } p \text{ is odd.} \end{cases}$$

Proof. The necessity of connectedness of a country is implied by the fact that the distances of citizens from the government used in the following expressions for the heterogeneity of a country are obtained when $\mathcal{P}$ is connected and they are lower bounds for the distances in a country that is not connected.

Case $p = 2k$. Let the government be located in the segment between the two middle citizens, at distance $x \in \langle 0, 1\rangle$ from the left one of them. Then

$$h(2k) = \sum_{m=1}^{k}(m-1+x) + \sum_{m=1}^{k}(m-x) = k(k+1) - k = k^2 = \frac{p^2}{4}.$$

For $p = 2k+1$ we have

$$h(2k+1) = 2\sum_{m=1}^{k} m = k(k+1) = \frac{p^2-1}{4}.$$

□

In an efficient configuration the heterogeneity of each country $\mathcal{P}$ with p citizens must be equal to $h(p)$. Now we shall prove that the sequence $h(p)$, $p = 1, 2, \ldots$ is convex, which will ensure that in an efficient configuration the sizes of countries are approximately equal.

LEMMA 3. *For each $p \geq 2$ and q, $0 < q < p$ we have*

$$h(p+q) + h(p-q) \geq 2h(p),$$

while equality occurs only if p is even and $q = 1$.

Proof. From Lemma 2 we have

$$\begin{aligned} h(p+q) + h(p-q) &\geq \frac{(p+q)^2-1}{4} + \frac{(p-q)^2-1}{4} = \frac{p^2+q^2-1}{2} \\ &\geq \frac{p^2}{2} \geq 2h(p). \end{aligned}$$

The first inequality is fulfilled as equality if and only if both $p+q$ and $p-q$ are odd; the second one is equality if and only if $q = 1$ and the third one if and only if p is even. Hence the desired result follows. □

THEOREM 1. *In an efficient configuration $\mathcal{E}$ the sizes of two countries may differ by at most* 2, *and if this occurs, the maximum and the minimum sizes of a country in $\mathcal{E}$ are odd.*

In general, it is now easy to see how to generate an efficient configuration for the world population of W citizens and a fixed number $N \leq W$ of countries. First of all, we divide W by N. If $W \ mod \ N = M$ is zero, then all the countries will have the same size $\frac{W}{N}$. Otherwise M is nonzero but smaller than N, hence we will have M countries of size $\lceil \frac{W}{N} \rceil$ and $N - M$ countries of size $\lfloor \frac{W}{N} \rfloor$. Configurations obtained in this way will be called *standard configurations* and for given W and N they will be denoted by $\mathcal{E}(W, N)$.

Moreover, if M is at least 2 and $\lceil \frac{W}{N} \rceil$ is even or if $N - M$ is at least 2 and $\lfloor \frac{W}{N} \rfloor$ is even, we can choose several pairs of countries with equal even size $q = \lceil \frac{W}{N} \rceil$ or $q = \lfloor \frac{W}{N} \rfloor$, replace each such pair by two countries with sizes $q - 1$ and $q + 1$ and obtain another efficient configuration.

Example. If $W = 6$ and $N = 3$, the sizes of countries in an efficient configuration are (2,2,2) or (3,2,1); for $W = 8$ and $N = 3$ the only possible efficient configurations are (3,3,2) and its permutations.

Now consider $W = 100$, $N = 8$. Here $\lfloor \frac{100}{8} \rfloor = 12$ and $100 \ mod\ 8 = 4$, hence

$$\mathcal{E}(100, 8) = (13, 13, 13, 13, 12, 12, 12, 12), \tag{3.4}$$

other possibilities are

$$(13, 13, 13, 13, 11, 13, 12, 12) \text{ and } (13, 13, 13, 13, 11, 13, 11, 13), \tag{3.5}$$

so all the efficient configurations for $W = 100$ and $N = 8$ ca be expressed as permutations of (3.4) and (3.5).

It is clear that for a given W and each value of parameter c there exists at least one efficient configuration. In what follows we shall describe an approach for finding the efficient configurations for given W and c with the number of countries not prescribed in advance .

The total cost of the standard configuration $\mathcal{E}(W, N)$ is a linear function of c

$$C_{W,N}(c) = N + H(\mathcal{E}(W, N))c. \tag{3.6}$$

To find the efficient number of countries N^* for a given c, we have to find the global minimum of functions of the form (3.6) for $N = 1, 2, \dots, W$. Fortunately, it is not necessary to compare $C_{W,N^*}(c)$ with $C_{W,N}(c)$ for all values of N, as we will just prove. We shall denote by $c_N^M(W)$ for $M < N$ the point where the functions $C_{W,M}(c)$ and $C_{W,N}(c)$ intersect. To begin with, we first compute the intersection $c_{N+1}^N(W)$ of two 'adjacent' costs, which has to fulfill

$$N + H(\mathcal{E}(W, N)).c_{N+1}^N(W) = N + 1 + H(\mathcal{E}(W, N + 1)).c_{N+1}^N(W). \tag{3.7}$$

From Equation 3.7 we get

$$c_{N+1}^N(W) = \frac{1}{H(\mathcal{E}(W, N)) - H(\mathcal{E}(W, N + 1))} \tag{3.8}$$

LEMMA 4. *The sequence* $c_1^0(W) = 0, c_2^1(W), c_3^2(W) \dots, c_W^{W-1}(W), c_{W+1}^W(W) = 1$ *defined by (3.8) is nondecreasing for each* W.

Proof. We need to show that $c_N^{N-1}(W) \leq c_{N+1}^{N}(W)$ for all $N = 1, 2, \ldots, W-1$, which is equivalent, since $H(\mathcal{E}(W,N)) - H(\mathcal{E}(W,N+1)) > 0$ for all N, to

$$H(\mathcal{E}(W,N)) + H(\mathcal{E}(W,N)) \leq H(\mathcal{E}(W,N-1)) + H(\mathcal{E}(W,N+1)). \tag{3.9}$$

Now both sides of (3.9) correspond to the heterogeneity of some configuration with $2N$ countries of the world population $2W$; but in the left hand side we have in fact the heterogeneity of an efficient configuration $\mathcal{E}(2W, 2N)$ with $2N$ countries and in the right hand side we have heterogeneity of some other configuration for $2W$ citizens. Therefore the desired inequality follows. □

THEOREM 2. *For given W and $c \in \langle 0, 1\rangle$, an efficient configuration exists and it has $N^* \leq W$ countries if and only if $c \in \langle c_{N^*}^{N^*-1}(W), c_{N^*+1}^{N^*}(W)\rangle$. If $c > 1$ then there is a unique efficient configuration having W one-citizen countries.*

Proof. Since $C_{W,1}(0) = 1 < C_{W,2}(0) = 2 < \ldots < C_{W,W}(0) = W$ and $C_{W,1}(1) \geq C_{W,2}(1) \geq \cdots \geq C_{W,W}(1)$, we immediately have that $0 < c_{N+1}^{N}(W) \leq 1$ and it remains to show that

$$(\forall N < N^*) c_{N^*}^{N}(W) \leq c_{N^*}^{N^*-1}(W)$$

and

$$(\forall N > N^*) c_{N}^{N^*}(W) \geq c_{N^*+1}^{N^*}(W).$$

We shall show the first inequality, the proof of the second one is similar.

Due to Lemma 4 we have $c_{N^*-1}^{N^*-2}(W) \leq c_{N^*}^{N^*-1}(W)$. If this inequality is fulfilled strictly, we have for $c = c_{N^*-1}^{N^*-2}(W)$ that $C_{W,N^*}(c) > C_{W,N^*-1}(c) = C_{W,N^*-2}(c)$ and since the rate of growth of $C_{W,N^*-2}(c)$ is greater than that of $C_{W,N^*-1}(c)$, the former function must intersect $C_{W,N^*}(c)$ earlier than the latter. (See Figure 1

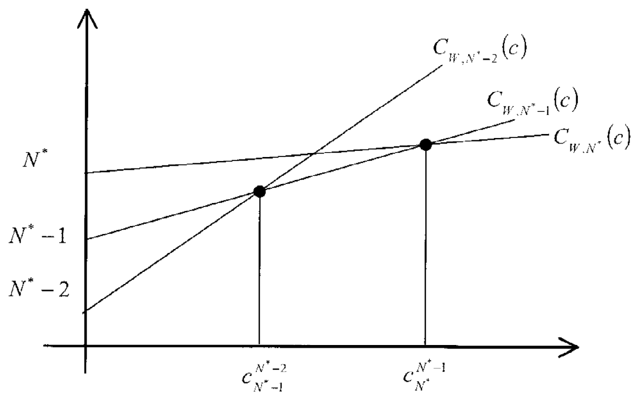

Figure 1.

for illustration.) Hence we get $c_{N^*}^{N^*-2}(W) \leq c_{N^*}^{N^*-1}(W)$ and the desired inequality follows by induction. □

To simplify the search for efficient configuration even further, we have the following assertions.

LEMMA 5. *Let W and N be such that $\lceil \frac{W}{N} \rceil \leq 3$. Then $N + H(\mathcal{E}(W, N)) = W + H(\mathcal{E}(W, W))$.*

Proof. If $\lceil \frac{W}{N} \rceil \leq 3$ then the only possible sizes of countries in $\mathcal{E}(W, N)$ are 1,2 and 3. Let us denote their numbers n_1, n_2 and n_3 respectively. Then $C_{W,N}(c) = n_1 + n_2 + n_3 + (n_1.0 + n_2.1 + n_3.2)c$, hence $N + H(\mathcal{E}(W, N)) = n_1 + 2n_2 + 3n_3 = W = W + H(\mathcal{E}(W, W))$. □

COROLLARY 1. *For a given W and N such that $N < W$ and $\lceil \frac{W}{N} \rceil \leq 3$, if there exists $N' < N$ such that $\lceil \frac{W}{N'} \rceil \leq 3$, then $\mathcal{E}(W, N)$ is efficient if and only if $c = 1$.*

For illustration, we consider $W = 24$. Table I gives the efficient structures and Figure 2 depicts the cost functions $C_{24,N}(c)$ for various N. Corollary 1 ensures that it is not necessary to consider $N > 8$, since configurations $\mathcal{E}(24, N)$ for $N = 9, 10, \ldots, 23$ are efficient if and only if $c = 1$ and $\mathcal{E}(24, 24)$ is efficient for all $c \geq 1$.

Table I. Efficient configurations for $W = 24$.

N	Efficient configurations	$C_{W,N}(c)$	Global optimum for
1	(24)	$1 + 144c$	$c \in \langle 0, \frac{1}{72} \rangle$
2	(12,12), (11,13)	$2 + 72c$	$c \in \langle \frac{1}{72}, \frac{1}{24} \rangle$
3	(8,8,8), (7,9,8)	$3 + 48c$	$c \in \langle \frac{1}{24}, \frac{1}{12} \rangle$
4	(6,6,6,6), (5,7,6,6), (5,7,5,7)	$4 + 36c$	$c \in \langle \frac{1}{12}, \frac{1}{8} \rangle$
5	(5,5,5,5,4)	$5 + 28c$	$c \in \langle \frac{1}{8}, \frac{1}{4} \rangle$
6	(4,4,4,4,4,4), (3,5,4,4,4,4)	$6 + 24c$	$c = \frac{1}{4}$
	(3,5,3,5,4,4), (3,5,3,5,3,5)		
7	(4,4,4,3,3,3,3), (4,5,3,3,3,3,3)	$7 + 20c$	$c = \frac{1}{4}$
8	(3,3,3,3,3,3,3,3)	$8 + 16c$	$c \in \langle \frac{1}{4}, 1 \rangle$

4. Stable configurations

The notion of stability is based on an assumption that an individual can leave his country and/or join another country without obtaining the consent of any of the countries affected (see Tiebout, 1956; Westhof, 1977; Haeringer, 2000). We suppose that the position of the government in a country is always in the middle

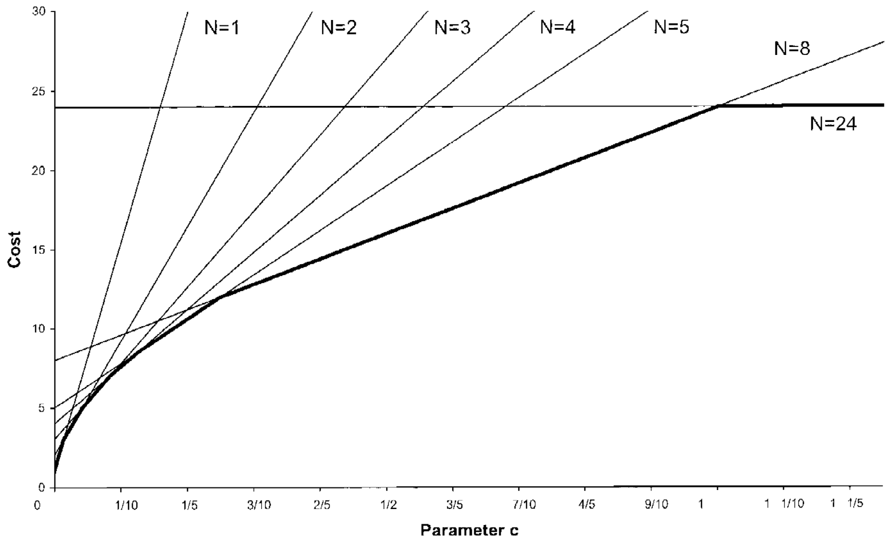

Figure 2.

of the country (which is one of the efficient locations of the government for that country, as proved in Lemma 1). In case a change in the configuration occurs, the location of the government in each of the affected countries will move to recover the minimal possible heterogeneity (this may be justified by voting of the citizens over the location of the government, see Alesina and Spolaore (1997) for a justification of this assumption). Formally we shall define a stable configuration as follows:

DEFINITION 1. *A configuration $\mathcal{E}$ of countries is said to be stable if none of the players can obtain a higher utility when leaving his country in $\mathcal{E}$ and either*

1. *creating a country of his own or*
2. *joining a neighbouring country.*

Haeringer (2000) proved in a more general model (with a monotone distribution of individuals on the line, identical concave utility function for each citizen and under the rule that an individual can join any country he wishes) the following assertion:

LEMMA 6. *If a configuration $\mathcal{E}$ is stable then all the countries in $\mathcal{E}$ are connected.*

In our model, increasing utility is equivalent to decreasing cost. Moreover, it is easy to see that the citizens with the highest cost in a country are its border citizens. So the necessary conditions for a stable configuration can be summarized in the following two assertions.

LEMMA 7. *Let $\mathcal{P}$ be a connected country with $p > 1$ citizens. Then no citizen of $\mathcal{P}$ wants to separate and create a country of his own if and only if*

$$c \leq \frac{2}{p}. \tag{4.10}$$

Proof. We have to compare the cost of a border citizen of the original country with the cost of a citizen in a one-citizen country, i.e. $\frac{1}{p} + \frac{p-1}{2}c$ should not be higher than 1. Suitable algebraic rearrangements yield the desired result. □

LEMMA 8. *Let two neighbouring countries, $\mathcal{P}$ and $\mathcal{Q}$, have sizes p and q, $p \leq q$ respectively. Then*

1. *the border citizen of $\mathcal{Q}$, who is closest to $\mathcal{P}$, does not want to jump to $\mathcal{P}$ if and only if $q = p + 1$; in case $q > p + 1$ it must hold*

$$c \leq \frac{2}{(p+1)q}; \tag{4.11}$$

2. *the border citizen of $\mathcal{P}$, who is closest to $\mathcal{Q}$ does not want to jump to $\mathcal{Q}$ if and only if*

$$c \geq \frac{2}{p(q+1)}. \tag{4.12}$$

Proof.

1. Consider the border citizen from the bigger country, $\mathcal{Q}$. If he prefers to stay in his original country than to join the smaller country, then his old cost is lower than his cost in the smaller country which will have $p + 1$ citizens and its government located in its middle:

$$\frac{1}{q} + \frac{q-1}{2}c \leq \frac{1}{p+1} + \frac{p}{2}c,$$

The above inequality is equivalent to

$$\frac{q-p-1}{2}c \leq \frac{q-p-1}{(p+1)q}.$$

Clearly, if $q = p + 1$, this inequality is trivially fulfilled and further manipulations give the desired inequality.

2. If the border citizen from the smaller country $\mathcal{P}$ with $p > 1$ citizens prefers to stay in $\mathcal{P}$ rather than to join $\mathcal{Q}$, then the following inequality must be fulfilled:

$$\frac{1}{p} + \frac{p-1}{2}c \leq \frac{1}{q+1} + \frac{q}{2}c,$$

which is equivalent to

$$\frac{p-q-1}{2}c \leq \frac{p-q-1}{p(q+1)}.$$

Since $p < q + 1$, after dividing by $p - q - 1$ we get the desired inequality. (If $p = 1$, the starting inequality $1 \leq \frac{1}{q+1} + \frac{q}{2}c$ also leads to the desired claim.) □

COROLLARY 2. *A configuration containing two neighbouring countries of sizes p and $q \geq p + 2$ can never be stable.*

Proof. According to Lemma 8, instability on the border of two neighbouring countries of sizes p and $q \geq p + 2$ will not occur if and only if

$$\frac{2}{p(q+1)} \leq c \leq \frac{2}{(p+1)q},$$

but since $(p + 1)q > p(q + 1)$ holds for each pair of integers p, q with $q > p$, such c can never exist. □

Since in a configuration with just one country Lemma 8 is irrelevant, Lemma 7 implies:

COROLLARY 3. *For a given world population W and a value of parameter c, the configuration $\mathcal{E} = (W)$ is stable if and only if $W.c \leq 2$.*

COROLLARY 4. *The configuration in which all countries have size 1 is the only stable configuration for $c > 1$. Conversely, this configuration cannot be stable if $c < 1$.*

Proof. If $c > 1$ and there exists a country with size $p \geq 2$ in a configuration, then the inequality $c \leq \frac{2}{p}$ cannot be fulfilled. Conversely, if in a configuration $\mathcal{E}$ all countries have size 1, then the only constraint that applies is the converse of inequality (4.12), which in this case implies $c \geq 1$. □

COROLLARY 5. *A configuration $\mathcal{E}$ containing at least two countries, but all with the same size $p > 1$ is stable if and only if*

$$\frac{2}{p(p+1)} \leq c \leq \frac{2}{p}.$$

COROLLARY 6. *A configuration $\mathcal{E}$ containing countries with sizes $p_1 < p_2 < \cdots < p_k$ for $k \geq 2$, but such that the size difference of two neighbouring countries is never more than 1 and there are no neighbouring countries with size p_1 is stable if and only if*

$$\frac{2}{p_1(p_1+2)} \leq c \leq \frac{2}{p_k}.$$

If there exist neighbouring countries with size p_1 then p_1 has to fulfill the stronger condition

$$\frac{2}{p_1(p_1+1)} \leq c.$$

We shall interpret the statements of Corollaries 5 and 6 from a different angle, namely enabling us to compute the possible country sizes for a given c. Let us therefore denote

$$t_1^L = \sqrt{\frac{2}{c} + \frac{1}{4}} - \frac{1}{2} \tag{4.13}$$

$$t_2^L = \sqrt{\frac{2}{c} + 1} - 1 \tag{4.14}$$

$$t^U = \frac{2}{c} \tag{4.15}$$

and Corollary 6 can now be reformulated as:

THEOREM 3. *For a given value of parameter $c \leq 1$, the size p of any country in a stable configuration $\mathcal{E}$ with at least two countries must fulfill $p \in \langle t_1^L, t^U \rangle$; if there are no neighbouring countries both with their sizes equal to the minimum size of a country in $\mathcal{E}$, then the lower bound has to be weakened to t_2^L.*

A simple algebraic procedure will show that $t_1^L \leq t^U - 1$ for each $c \leq 1$, which means that there is always an integer p in interval $\langle t_1^L, t^U \rangle$; however, this does not automatically imply that for each c and each W there will always exist a stable configuration.

THEOREM 4. *For each W and $c \leq 1$ a stable configuration exists, namely it is the configuration $\mathcal{E}(W, N)$ for suitable N.*

Proof. If $W \leq t^U = \frac{2}{c}$ then configuration (W) is stable. So we have to deal with the case $W > \frac{2}{c}$ and for each value of such W and $c < 1$ we construct a stable configuration.

For $c \in \langle \frac{2}{3}, 1)$ we have $t_2^L \leq 1$ and country sizes 1 and 2 are possible. So for W even we set $\mathcal{E} = (2, 2, \ldots, 2)$ and for W odd we shall have $\mathcal{E} = (1, 2, \ldots, 2)$. For $c \in \langle \frac{1}{3}, \frac{2}{3} \rangle$ we have $t_1^L \leq 2$ and $t^U \geq 3$, so for W even we have again a configuration consisting of solely two-citizen countries and for W odd we have $\mathcal{E} = (3, 2, \ldots, 2)$ (remember, that now $W > 3$). If $c \leq \frac{1}{3}$, then there are always at least two feasible country sizes if and only if $t_1^L \leq t^U - 2$, which is equivalent to the quadratic inequality $c^2 - 4c + 2 \geq 0$ with the solution set equal to $(-\infty, 2 - \sqrt{2} \rangle \cup \langle 2 + \sqrt{2}, \infty)$. This solution set contains the interval $(0, \frac{1}{3} \rangle$. So suppose that we choose for the size of a country $k = \lceil t_1^L \rceil$, so $k + 1$ is also feasible. Now consider the divisibility of W by k. If $W \bmod k = 0$, we set $\mathcal{E} = (k, k, \ldots, k)$; if $W \bmod k = 1$, there will be just one country of size $k + 1$ in $\mathcal{E}$, the rest will have size k. In general, if $W \bmod k = r$ for some $r < k$, $\mathcal{E}$ will contain r countries of size $k + 1$ and the rest will have size k.

This construction will be possible for all values of k and r if W is sufficiently large, i.e. at least equal to $(k-1)(k+1) = k^2 - 1$. Due to our choice of k we know that $k \leq t_1^L + 1$ and since $W \geq \frac{2}{c}$, we must have $(t_1^L + 1)^2 - 1 \leq \frac{2}{c}$, which trivially holds. □

The following example shows that the set of stable configurations may be quite diverse for some values of parameters.

Example. If $c = \frac{1}{4}$ then $p \in \{3, 4, 5, 6, 7, 8\}$, or $p \in \{2, 3, 4, 5, 6, 7, 8\}$ if there are no neighbouring countries with size 2.

If $W = 5$ then the sizes $6, 7, 8$ are irrelevant, so we get that the only stable configurations are (5) and $(2, 3)$. However, if $W = 24$, the number of stable configurations is already very high. All of them, apart from suitable permutations of countries, are listed in Table II, sorted according to the number of countries N. Those, that are also efficient for $c = \frac{1}{4}$, are marked with a star. On the other hand, there exist configurations efficient for $c = \frac{1}{4}$, which cannot be stable, even with a suitable permutation of countries, for example $(3, 3, 3, 5, 5, 5)$.

Table II. Stable configurations for $c = \frac{1}{4}$ and $W = 24$.

N	Stable configurations
3	(8, 8, 8)
4	(6, 6, 6, 6), (5, 6, 6, 7)
5	(4, 5, 5, 5, 5)*, (4, 4, 5, 5, 6), (3, 4, 5, 6, 6)
6	(4, 4, 4, 4, 4, 4)*, (3, 4, 4, 4, 4, 5)*, (3, 3, 4, 4, 5, 5)*, (2, 3, 4, 5, 5, 5), (2, 3, 4, 4, 5, 6)
7	(3, 3, 3, 3, 4, 4, 4)*, (2, 3, 3, 3, 4, 4, 5), (2, 3, 2, 3, 4, 5, 5), (3, 3, 3, 3, 3, 4, 5)*
8	(3, 3, 3, 3, 3, 3, 3, 3)*, (2, 3, 3, 3, 3, 3, 3, 4), (2, 3, 2, 3, 3, 3, 4, 4)
9	(2, 3, 2, 3, 2, 3, 2, 3, 4)

5. Group Deviations

Since the set of stable configurations may be so large, we try to see what happens, if we allow for group deviations. The simplest possibility is that two people, who originally were not in the same country, leave their countries and form a new country. Now again we suppose that the original countries cannot prevent the secession, but both deviating citizens must strictly increase their utilities (or, equivalently, strictly decrease their cost).

DEFINITION 2. *A configuration is called strongly stable if it is stable and no two citizens from two different neighbouring countries want to leave their original countries and form a country of their own.*

If a configuration contains just one country, then only Lemma 7 applies and we have similarly as in Section 4:

COROLLARY 7. *For a given world population W and a value of parameter c, the configuration $\mathcal{E} = (W)$ is strongly stable if and only if $W.c \leq 2$.*

Clearly, no citizen of a two-citizen country will be willing to participate in a deviation according to Definition 2. On the other hand, a deviation of this sort brings the greatest improvement compared to their original situation to a pair of neighbouring citizens. Therefore it is sufficient to consider only the following case:

LEMMA 9. *Let $\mathcal{E}$ be a stable configuration and $\mathcal{P}$ and $\mathcal{Q}$ two neigbouring countries in $\mathcal{E}$ with sizes p and q, $p \leq q$ respectively. Then the two neighbours, one from $\mathcal{P}$ and the other from $\mathcal{Q}$ will not unanimously want to leave $\mathcal{P}$ and $\mathcal{Q}$ respectively and form a country of their own if and only if one of the following cases occurs:*

1. *$p = q = 1$ and $c \geq 1$,*
2. *$p = 2$ or $q = 2$;*
3. *$2 < p$ and $c \leq \frac{1}{p}$.*

Proof. A border citizen from a country with $p > 1$ citizens prefers staying in his original country before creating a two-citizens country if and only if

$$\frac{1}{p} + \frac{p-1}{2}c \leq \frac{1}{2} + \frac{1}{2}c,$$

which is equivalent to

$$\frac{p-2}{2}c \leq \frac{p-2}{2p}.$$

This inequality is trivialy fulfilled if $p = 2$; if $p > 2$ we get $c \leq \frac{1}{p}$. On the other hand, the 'nondeviation' inequality for a citizen from a one-citizen country is $1 \leq \frac{1}{2} + \frac{1}{2}c$, equivalent to $c \geq 1$. □

COROLLARY 8. *For a given c, the size $p > 2$ of a country in a strongly stable configuration which has a neighbouring country with size $q \geq p$ cannot exceed $\frac{1}{c}$.*

Now we can summarize the conditions for the country sizes in a strongly stable configuration in the following assertion:

LEMMA 10. *For a given value of parameter c, the configuration $(1, 1, \ldots, 1)$ is strongly stable if and only if $c \geq 1$. For $c < 1$ in a strongly stable configuration $\mathcal{E}$ with at least two countries each country $\mathcal{P}$ either must have a neighbour of size 2 or $p \geq t_1^L$ if $\mathcal{P}$ has no neighbour of size p, otherwise it is sufficient to have $p \geq t_2^L$; and $p \leq \frac{1}{c} + 1$ if $\mathcal{P}$ has no neighbour with size exceeding $\frac{1}{c}$, otherwise $p \leq \frac{1}{c}$.*

THEOREM 5. *For each value of W and c there exists a strongly stable configuration.*

Proof. If $c \geq 1$ then the configuration containing only countries of size 1 is strongly stable. For $c < 1$ we can repeat the construction in the proof of Theorem 4. For $c \geq \frac{1}{3}$ in the constructed configuration each country has a neighbouring country of size 2, so the construction gives a strongly stable configuration. For $c < \frac{1}{3}$ we need to ensure that $k+1 \leq \frac{1}{c}$, which means that we need $t_1^L + 2 \leq \frac{1}{c}$, which is equivalent to the quadratic inequality $2c^2 - 5c + 1 \geq 0$. Its solution set $(-\infty, \frac{5-\sqrt{17}}{4}\rangle \cup \langle \frac{5+\sqrt{17}}{4}, \infty)$ however does not include the whole interval $(0, \frac{1}{3}\rangle$, but at least $(0, \frac{1}{5}\rangle$ is covered by this case.

For $c \in \langle \frac{1}{4}, \frac{1}{3}\rangle$ we have $2 \leq t_1^L \leq 3 \leq \frac{1}{c}$ and $t_2^L \leq 2$, so 2 is also an admissible country size, if no neighbours of size 2 exist. So we can take configuration $(3, 3, \ldots, 3)$ if $W \equiv 0(mod\ 3)$ and for cases $W \equiv 1(mod\ 3)$ and $W \equiv 2(mod\ 3)$ configurations $(2, 3, \ldots, 3, 2)$ and $(2, 3, \ldots, 3)$ respectively. This will be possible for all W that fulfill $W \geq \frac{2}{c} \geq 6$.

For $c \in \langle \frac{1}{5}, \frac{1}{4})$ the smallest possible country size is 3, but now $\frac{1}{c} \geq 4$, so 4 as the size of a country is possible. So we can take configuration $(3, 3, \ldots, 3)$, $(3, 3, \ldots, 3, 4)$ and $(4, 3, \ldots, 3, 4)$ for if $W \equiv 0, 1, 2(mod\ 3)$ respectively. Here again W is sufficiently large, since $W \geq \frac{2}{c} > 8$. □

However, the set of strongly stable configurations is smaller than that of stable configurations. Again for $c = \frac{1}{4}$, the feasible intervals of country sizes are $\{3, 4\}$ and $\{2, 3, 4\}$ if there are no neighbouring countries with sizes 2. However, countries of size 5 are also admissible, as long as all their neighbours have size at most 4. All the possible configurations for $W = 24$ are summarized in Table III.

Table III. Strongly stable configurations for $c = \frac{1}{4}$ and $W = 24$.

N	Strongly stable configurations
6	(4, 4, 4, 4, 4, 4)*, (3, 4, 4, 4, 4, 5)*, (3, 3, 4, 5, 4, 5)*
7	(3, 3, 3, 4, 4, 4, 4)*, (2, 3, 3, 3, 4, 4, 5), (3, 3, 3, 3, 3, 4, 5)*
8	(3, 3, 3, 3, 3, 3, 3, 3)*, (2, 3, 3, 3, 3, 3, 3, 4), (2, 3, 2, 3, 3, 3, 4, 4)
9	(2, 3, 2, 3, 2, 3, 2, 3, 4)

6. Conclusion and Directions for Further Research

In this paper we studied a simple one-dimensional model of country formation. We considered optimality as well as stability critera and have shown that configurations that arise in the two cases may be quite different.

For each parameter value of a particular instance of our problem, the set of efficient configurations is always nonempty and its structure is quite simple. On the other hand, although stable configurations always exists, they are very diverse if we only allow individual moves, and are still quite variable if the simplest possibility of group deviations is allowed.

It would be interesting to see how these results change if some of the characteristics of the model are modified, for example, we may have:

- the distances between two neigbours arbitrary, and/or
- in each discrete point an arbitrary finite population size, and/or
- secessions of larger groups of citizens allowed, and/or
- possible only if the majority of citizens in affected countries approve them,
- different forms of utility functions.

We expect that some algorithms for obtaining efficient and stable configurations in each case could be derived, possibly using methods of discrete location theory, described in [6] and [17].

References

1. Alesina, A. and Spolaore, E. (1997), On the Number and Size of Nations, *The Quarterly Journal of Economics*, November 1997: 1027–1056.
2. Bewley, T.F. (1981), A Critique of Tiebout's Theory of Local Public Expenditures, Econometrica 49: 713–741.
3. Bogomolnaia, A. and Jackson, M. O. (1998), *The Stability of Hedonic Coalition Structures*, mimeo CalTech, draft November 13, .
4. Bolton, G. and Roland, G. (1997), The Breakup of Nations: A Political Economy Analysis, Quarterly Journal of Economics, CXII, 1057–1089.
5. Bolton, G., Roland G. and Spolaore, E. (1996), *Economic Theories of the Integration and Break-up of Nations*, European Economic Review, Papers and Proceedings, XXXIX, 697–705.
6. Christofides, N. (1995), *Graph Theory, An Algorithmic Approach*, Academic Press, London.
7. Dahm, M. (1999), *On the Number and Size of Nations Revisited: The Trade-Off between Efficiency and Stability*, Master Thesis, Universitat Autonoma de Barcelona, July 23.
8. Dahm, M. Stability and Efficiency in a Discrete Model of Country Formation with an Unidimensional policy Space, manuscript, 2000.
9. Demange, G. (1994), Intermediate preferences and stable coalitions structures, *Journal of Mathematical Economics* 23: 45–58.
10. Drèze, J.H. and Greenberg, J. (1980), Hedonic Coalitions: Optimality and Stability, Econometrica, Vol. 48(4): 987–1003.
11. Greenberg, J. and Weber, S. (1986), Strong Tiebout Equilibrium under restricted preferences domain, *Journal of Economic Theory* 38, 101–117.
12. Greenberg, J. and Weber, S. (1993), Stable coalition structures with an unidimensional set of alternatives, Journal of Economic Theory 60: 62–82.
13. Gusnerie, R. and Oddou, C. (1981), Second best taxation as a game, *Journal of Economic Theory* 25, 67–91.
14. Haeringer, G. Stable coalition structures with fixed decision scheme, mimeo, 2000.
15. Jehiel and Scotchmer, (1997), *Constitutional rules of exclusion in jurisdiction formation*, Working paper, University of California, Berkeley.
16. Le Breton, M. and Weber, S. The Art of Making Everybody Happy: How to prevent a Secession?, mimeo, December 2000.
17. Mirchandani, P.B. The p-Median Problem and Generalizations, in *Discrete Location Theory*.

18. Tiebout, C. (1956), A Pure Theory of Local Expenditure, *Journal of Political Economy*, LXIV: 416–424.
19. Westhoff, F. (1977), Existence of equilibria in economies with a local public good, *JET* 14: 84–112.